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Non-linear sigma model on the fuzzy supersphere

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Abstract: In this note we develop fuzzy versions of the supersymmetric non-linear sigma model on the supersphere $S^{(2,2)}$. In [1] Bott projectors have been used to obtain the fuzzy $\mathbb{C}P^1$ model. Our approach utilizes the use of supersymmetric extensions of these projectors. Here we obtain these (super)-projectors and quantize them in a fashion similar to the one given in [1]. We discuss the interpretation of the resulting model as a finite dimensional matrix model.

Keywords: Sigma Models, Superspaces, Non-Commutative Geometry, Matrix Models.

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1. Introduction

In past few years studies of field theories on non-commutative manifolds have been very fruitful. To construct such theories one usually starts with a continuum theory on a manifold $\mathcal{M}$ and replaces the commutative algebra $\mathcal{A}$ of functions on $\mathcal{M}$ by a non-commutative algebra A which preserves most of the symmetries of the continuum theory and which approximates the commutative algebra $\mathcal{A}$ and hence the continuum theory in the commutative limit. In the literature "*fuzzy manifold(s)*" are usually defined as the class of non-commutative manifolds where the non-commutative algebra $\mathcal{A}$ has a sequence of finite-dimensional irreducible representations with ever-increasing dimensions and tending to the commutative algebra in the infinite dimensional limit. It is possible to realize field theories on fuzzy manifolds as finite dimensional matrix models. Field theories on the non-commutative (fuzzy) sphere S_F^2 and the fuzzy supersphere $S_F^{(2,2)}$ are two such examples. S_F^2 is based on the irreducible representations of the su(2) Lie algebra, whereas the $S_F^{(2,2)}$ is described by the irreducible representations of the Lie superalgebra osp$(2,1)$. To date many studies on different and novel aspects of field theories on S_F^2 and $S_F^{(2,2)}$ have been carried out [2]–[8].

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Recently, $\mathbb{C}P^1$ model on the fuzzy sphere S_F^2 have been studied from several different points of view [9, 10, 1]. In [1] the commutative theory have been reformulated by replacing the non-linear fields with a certain class of projectors called "Bott Projectors". A discrete (fuzzy) version of these projectors can be easily obtained and they have permitted the construction of a fuzzy $\mathbb{C}P^1$ model in a rather straightforward way.

In this paper we address the question of constructing a fuzzy supersymmetric non-linear sigma model on $S^{(2,2)}$. For this purpose we obtain the supersymmetric extensions of the Bott projectors and quantize them in a similar manner as discussed in [1]. Using the quantized (super)-projectors and the already known description of $S^{(2,2)}$ in terms of the Lie superalgebras osp$(2,1)$ and osp$(2,2)$ and their associated Lie supergroups we construct the fuzzy supersymmetric non-linear sigma model on $S^{(2,2)}$. We interpret the resulting theory as a finite dimensional matrix model and comment on its various physical properties.

2. $\mathbb{C}P^1$ sigma model and Bott projectors

Non-linear sigma models are customarily defined in terms of a field that maps the world-sheet to the target manifold. In the case of the $\mathbb{C}P^1$ models both world-sheet and the target manifolds are 2-spheres (S^2) and the field $\vec{n}$ maps the point x of the world-sheet

$$S^2 = \langle x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \; x_i x_i = 1 \rangle \tag{2.1}$$

to a point on the target manifold

$$S^2 = \langle \vec{n}(x) = (n_1(x), n_2(x), n_3(x)) \in \mathbb{R}^3 | \; n_a(x) n_a(x) = 1 \rangle \, . \tag{2.2}$$

As is well known these maps are classified in terms of an integer κ called the winding number since the second homotopy class $\pi_2(S^2) = \mathbb{Z}$.

An alternative formulation of $\mathbb{C}P^1$ model which happens to be more convenient for passage to fuzzy $\mathbb{C}P^1$ model have been considered in [1]. This formulation exploits the use of certain class of projectors, known as Bott projectors instead of the non-linear fields. At the topological sector $\kappa = 1$ the Bott projector can be expressed in terms of $\vec{n}$ as

$$P(x) = \frac{1 + \vec{\tau} \cdot \vec{n}(x)}{2} \tag{2.3}$$

where $\vec{\tau}$ are the Pauli matrices. $P(x)$ is a projector since $P^2(x) = P(x)$ and $P^\dagger(x) = P(x)$. At the topological sector κ, Bott projector can be expressed by introducing the partial isometries[1] $\vartheta_\kappa^\dagger$ [11]

$$\vartheta_\kappa^\dagger(z) = \left(\bar{z}_1^\kappa \quad \bar{z}_2^\kappa \right) \frac{1}{\sqrt{Z_\kappa}} \, , \quad \vartheta_\kappa(z) = \begin{pmatrix} z_1^\kappa \\ z_2^\kappa \end{pmatrix} \frac{1}{\sqrt{Z_\kappa}} \, , \qquad Z_\kappa = |z_1|^{2\kappa} + |z_2|^{2\kappa} \tag{2.4}$$

[1] To be more precise the partial isometry $\vartheta_\kappa^\dagger$ in the algebra $\mathcal{A} = C^\infty(S^3) \otimes Mat_{2\times 2}\mathbb{C}$ is the matrix $\left(\begin{smallmatrix} \bar{z}_1^\kappa & \bar{z}_2^\kappa \\ 0 & 0 \end{smallmatrix}\right)$. But for all practical calculations it is perfectly safe to call (2.4) as the partial isometry, thus we do so from now on.

where $z = (z_1, z_2)$ is a point on $S^3 = \langle z = (z_1, z_2) \in \mathbb{C}^2 |\; |z|^2 := |z_1|^2 + |z_2|^2 = 1\rangle$ and "bar" stands for complex conjugation. Using the Hopf fibration $\mathrm{U}(1) \to S^3 \to S^2$, points x on the world-sheet S^2 is expressed in terms of z as

$$x_i = z^\dagger \tau_i z\,. \tag{2.5}$$

By definition $\vartheta_\kappa^\dagger$ is a partial isometry if and only if $\vartheta_\kappa(z)\vartheta_\kappa^\dagger(z)$ is a projection. It is straightforward to check that $P_\kappa(x)$ in the topological sector κ is given as

$$P_\kappa(x) = \vartheta_\kappa(z)\vartheta_\kappa^\dagger(z) = \frac{1}{Z_\kappa}\begin{pmatrix} |z_1|^{2\kappa} & z_1^\kappa \bar{z}_2^\kappa \\ z_2^\kappa \bar{z}_1^\kappa & |z_2|^{2\kappa} \end{pmatrix} \tag{2.6}$$

is a projector: $P_\kappa(x)^2 = P_\kappa(x)\,, P_\kappa(x)^\dagger = P_\kappa(x)$.

The field $n_a^\kappa(x)$ is associated to $P_\kappa(x)$ by the formulas

$$n_a^\kappa(x) = \mathrm{Tr}\,\tau_a P_\kappa(x) = \vartheta_\kappa^\dagger(z)\tau_a\vartheta_\kappa(z)\,, \qquad P_\kappa(x) = \frac{1 + \vec{\tau}\cdot\vec{n}^\kappa(x)}{2}\,. \tag{2.7}$$

A phase change $z \to ze^{i\theta}$ induces the change $\vartheta_\kappa(z) \to \vartheta_\kappa(z)e^{i\kappa\theta}$. Nevertheless, this phase cancels in $\vartheta_\kappa(z)\vartheta_\kappa^\dagger(z)$ and $P_\kappa(x)$ is a function of x only.

In [1] an intuitive argument as well as an explicit calculation is given to show that κ appearing in equations (2.4) through (2.7) is indeed the winding number. Here we recollect the former. For $\kappa > 0$ consider the κ points (up to an overall phase of z which cancels out on x) of S^2 labeled by ℓ:

$$z_\ell = \left(z_1 e^{i\frac{2\Pi}{\kappa}\ell}\,, z_2\right) \qquad \ell \in (0\,, \kappa - 1)\,. \tag{2.8}$$

All z_ℓ map to the same point on the target manifold S^2 or equivalently, they all have the same projection via $P_\kappa(x)$, giving winding number κ.

It must be noticed that the form of $P_\kappa(x)$ is very particular. Nevertheless, the most general projector $\mathcal{P}_\kappa(x)$ can be obtained from

$$\mathcal{P}_\kappa(x) = U(x)P_\kappa(x)U(x)^\dagger \tag{2.9}$$

where $U(x) \in \mathrm{U}(2)$ is a 2×2 unitary matrix. The field associated to $\mathcal{P}_\kappa(x)$ is nothing but

$$n_a^{\kappa\prime}(x) = \mathrm{Tr}\,\tau_a\mathcal{P}_\kappa(x) \tag{2.10}$$

where $n_a^{\kappa\prime}(x) = R_{ab}n_b^\kappa(x)$, $U^\dagger\tau_a U = R_{ab}\tau_b$ and $R \in \mathrm{O}(3)$. The unitary transformation do not affect the the winding number since $\pi_2(\mathrm{U}(2)) = \{e\}$.

3. On the actions

A euclidean action in the κ-th topological sector is given in terms of the fields $n_a^\kappa(x)$ [2] by

$$S_\kappa = -\frac{1}{8\pi}\int_{S^2} d\Omega(\mathcal{L}_i n_a^\kappa)(\mathcal{L}_i n_a^\kappa) \tag{3.1}$$

[2] For brevity we drop the "prime" on the fields $n_a(x)$.

where $\mathcal{L}_i = -i(x \wedge \nabla)_i$ is the angular momentum operator and $d\Omega = d\cos\theta d\psi$. In terms of the projectors, S_κ can be expressed as

$$S_\kappa = -\frac{1}{4\pi}\int_{S^2} d\Omega\,\mathrm{Tr}\,(\mathcal{L}_i\mathcal{P}_\kappa)\,(\mathcal{L}_i\mathcal{P}_\kappa)\,. \tag{3.2}$$

The well known formulae for the winding number and BPS bound of this model can also be rewritten in terms of the projectors $\mathcal{P}_\kappa$. The actions given in (3.1) and (3.2) both do have discrete versions when the $\mathbb{C}P^1$ model is formulated on the fuzzy sphere S_F^2. However, it seems that the latter is better adapted for formulation of fuzzy $\mathbb{C}P^1$ sigma models; as will be discussed in section 6 it is possible to quantize the projectors in a straightforward manner. For a detailed discussion on the fuzzy $\mathbb{C}P^1$ model the reader is refered to [1].

In section 5 we develop the supersymmetric extension of the projectors $\mathcal{P}_\kappa(x)$ and apply this result to the description of non-linear sigma model first on the supersphere and then on the fuzzy supersphere. The latter will require the supersymmetric extension of quantized projectors.

4. The commutative and non-commutative (fuzzy) superspheres

4.1 The supersphere $S^{(2,2)}$

In this section we would like to collect some preliminary differential geometric and group theoretical formulae that is used to characterize the supersphere $S^{(2,2)}$ and its non-commutative (fuzzy) version $S_F^{2,2}$. The details of the very brief discussion below can be found in [7, 8].

The structure underlying the supersphere $S^{(2,2)}$ comes from the Lie superalgebras osp$(2,1)$ and osp$(2,2)$ and their associated Lie supergroups OSP$(2,1)$ and OSP$(2,2)$. osp$(2,1)$ is build up of the Lie algebra su(2) (even part) with generators $L_i\,,(i=1,2,3)$ and su(2) spinors $V_\alpha(\alpha=+,-)$ (odd part). osp$(2,2)$ Lie superalgebra is constructed by augmenting osp$(2,1)$ generators with an additional pair of spinors $D_\alpha(\alpha=+,-)$ and an additional even generator Γ. The graded commutation relations of osp$(2,2)$ generators read

$$\begin{aligned}
[L_i,L_j] &= i\epsilon_{ijk}L_k\,, & [L_i,V_\alpha] &= \frac{1}{2}(\sigma_i)_{\beta\alpha}V_\beta\,, & \{V_\alpha,V_\beta\} &= \frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i\,,\\
[L_i,\Gamma] &= 0\,, & [\Gamma,V_\alpha] &= D_\alpha\,, & [\Gamma,D_\alpha] &= V_\alpha\,,\\
[L_i,D_\alpha] &= \frac{1}{2}(\sigma_i)_{\beta\alpha}D_\beta\,, & \{D_\alpha,D_\beta\} &= -\frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i\,, & \{D_\alpha,V_\beta\} &= \frac{1}{4}C_{\alpha\beta}\Gamma\,.
\end{aligned} \tag{4.1}$$

where $i,j=1,2,3$, $\alpha,\beta=\pm$ and $C=i\sigma_2$. The graded commutation relations for the osp$(2,1)$ generators is given by the first line of (4.1).

In the corresponding enveloping algebras there are central polynomials - the Casimir operators in representations given by the formulas:

$$\begin{aligned}
K_2^{\mathrm{osp}(2,1)} &= L_iL_i + C_{\alpha\beta}V_\alpha V_\beta\,,\\
K_2^{\mathrm{osp}(2,2)} &= L_iL_i + C_{\alpha\beta}V_\alpha V_\beta - \left(C_{\alpha\beta}D_\alpha D_\beta + \frac{1}{4}\Gamma^2\right).
\end{aligned} \tag{4.2}$$

These Lie superalgebras are endowed with a grade dagger operation ‡ replacing the usual adjoint operation on the Lie algebras. Generators of osp$(2,2)$ fulfill the following reality conditions implemented by ‡:

$$L_i^\ddagger = L_i^\dagger = L_i\,,\qquad V_\alpha^\ddagger = C_{\alpha\beta}V_\beta\,,\qquad D_\alpha^\ddagger = -C_{\alpha\beta}D_\beta\,,\qquad \Gamma^\ddagger = \Gamma^\dagger = \Gamma\,. \tag{4.3}$$

The reality conditions fulfilled by osp$(2,1)$ is obtained by restricting to the relations fulfilled by L_i and V_α. The graded conjugation is extended to homogeneous elements A and B in enveloping algebras by

$$(AB)^\ddagger = (-1)^{|A||B|}B^\ddagger A^\ddagger\,. \tag{4.4}$$

Here $|A|$ and $|B|$ denote the degrees of A and B, respectively. By linearity the conjugation is extended to the whole enveloping algebra. The Casimir elements, given above, are real.

The supersphere $S^{(2,2)}$ is the adjoint orbit of the Lie supergroup OSP$(2,1)$. It can be obtained through a super generalization of the Hopf fibration for the 2-sphere. In the supersymmetric case this becomes $\mathrm{U}(1)\to S^{(3,2)}\to S^{(2,2)}$ where $S^{(3,2)}\equiv \mathrm{OSP}(2,1)$ and

$$S^{(2,2)} = S^{(3,2)}/\,\mathrm{U}(1)\,. \tag{4.5}$$

The superspace $\mathbb{R}^{(3,2)}$ is defined as the algebra of polynomials in generators x_i and θ_α satisfying reality conditions

$$x_i^\ddagger = x_i\,,\qquad \theta_\alpha^\ddagger = C_{\alpha\beta}\theta_\beta\,. \tag{4.6}$$

These conditions are extended as in (4.4) to all polynomials. The equation characterizing the adjoint orbit $S^{(2,2)}$ of osp$(2,1)$ is

$$S^{(2,2)} = \Big\langle (x_i\,,\theta_\alpha)\in\mathbb{R}^{(3,2)}\mid x_i^2 + C_{\alpha\beta}\,\theta_\alpha\theta_\beta = \frac{1}{4}\Big\rangle\,. \tag{4.7}$$

The action of osp$(2,1)$ on $S^{(2,2)}$ is the adjoint action and is given in terms of the differential operators

$$\begin{aligned}\ell_i &= -i\varepsilon_{ijk}x_j\partial_k - \frac{1}{2}(\sigma_i)_{\beta\alpha}\theta_\beta\partial_{\theta^\alpha}\\ v_\alpha &= -\frac{1}{2}(\sigma_i)_{\beta\alpha}\theta_\beta\partial_i + \frac{1}{2}(C\sigma_i)_{\alpha\beta}x_i\partial_{\theta^\beta}\,.\end{aligned} \tag{4.8}$$

corresponding to the osp$(2,1)$ generators L_i and V_α, respectively. It can be extended to an osp$(2,2)$ action which is not an adjoint action but it is closely related to it. (for details see [7, 8]). The additional differential operators have the form

$$\begin{aligned}d_\alpha &= -r\Big(1+\frac{2}{r^2}\Big)C_{\alpha\beta}\partial_{\theta^\beta} + \frac{1}{2r}(\sigma_i)_{\beta\alpha}\theta_\beta\mathcal{L}_i - \frac{\theta_\alpha}{2r}x_i\partial_i\\ \gamma &= \Big(\frac{\theta_+x_3}{r}+\frac{\theta_-x_+}{r}\Big)\partial_+ + \Big(\frac{\theta_+x_-}{r}-\frac{\theta_-x_3}{r}\Big)\equiv 2(\theta_-v_+-\theta_+v_-)\,.\end{aligned} \tag{4.9}$$

corresponding to the generators D_α and Γ of osp$(2,2)$.

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4.2 The fuzzy supersphere $S_F^{(2,2)}$

The fuzzy supersphere $S_F^{(2,2)}$ is obtained replacing $(x_i\,,\theta_\alpha) \in \mathbb{R}^{(3,2)}$ by suitable rescaled osp$(2,1)$ generators $X_i = \lambda L_i$ and $\Theta = \lambda V_\alpha$ with λ determined by the value of osp$(2,1)$ Casimir operator:

$$\frac{1}{4\lambda^2} = K_2^{\mathrm{osp}(2,1)}\,. \tag{4.10}$$

The fuzzy parameters then satisfy supersphere's defining relation

$$X_i X_i + C_{\alpha\beta}\Theta_\alpha\Theta_\beta = \frac{1}{4}\,. \tag{4.11}$$

The non-commutativity of the fuzzy supersphere follows from the graded commutation relations of X_i and Θ_α. For details we refer the reader to [7, 8].

5. Non-linear sigma model on $S^{(2,2)}$

5.1 Preliminaries

The superfield Φ on $S^{(2,2)}$ is a function of the variables $(x_i\,,\theta_\alpha)$; it is real provided that $\Phi^\ddagger = \Phi$. For a free real superfield multiplet the action is related to the osp$(2,1)$ invariant given as the difference of the quadratic Casimir operators:

$$K_2^{\mathrm{osp}(2,1)} - K_2^{\mathrm{osp}(2,2)} = C_{\alpha\beta}D_\alpha D_\beta + \frac{1}{4}\Gamma^2\,. \tag{5.1}$$

The action takes the form

$$S^{\mathrm{SUSY}} = \frac{1}{4\pi}\int d\mu\left(d_\alpha\Phi d_\alpha\Phi + \frac{1}{2}\gamma\Phi\frac{1}{2}\gamma\Phi\right) \tag{5.2}$$

where $d\mu = d^3x^i d\theta^+ d\theta^- \delta(x_i^2 + C_{\alpha\beta}\theta^\alpha\theta^\beta - \frac{1}{4})$, and d_α and γ are the differential operators given in (4.9).

For a free triplet real superfield $\Phi^a = \Phi^a(x_i\,,\theta_\alpha)$, $(a = 1, 2, 3)$, we just replace in Φ by Φ^a (with the summation over repeated index a understood). Now we define the O(3) sigma model [12] by putting on Φ^a the constraint

$$\Phi^a\Phi^a = 1 \qquad (a = 1, 2, 3)\,. \tag{5.3}$$

Then (5.2) and (5.3) defines the non-linear sigma model on the supersphere $S^{(2,2)}$ with the target manifold being S^2.

The superfield $\Phi^a(x_i\,,\theta_\alpha)$ can be expanded in powers of θ_α as

$$\Phi^a(x_i\,,\theta_\alpha) = n^a(x_i) + C_{\alpha\beta}\theta_\beta\psi_\alpha^a(x_i) + \frac{1}{2}F^a(x_i)C_{\alpha\beta}\theta_\alpha\theta_\beta \tag{5.4}$$

where $\psi^a(x_i)$ are two component Majorana spinors: $\psi_\alpha^{a\ddagger} = C_{\alpha\beta}\psi_\beta^a$, and $F^a(x_i)$ are auxiliary scalar fields. In terms of the component fields the constraint equation (5.3) splits to

$$n^a n^a = 1\,, \qquad n^a F^a = \frac{1}{2}\psi^{a\ddagger}\psi^a\,, \qquad n^a\psi_\alpha^a = 0\,. \tag{5.5}$$

5.2 Supersymmetric extensions of Bott projectors

A possible supersymmetric extension of the projector $\mathcal{P}_\kappa(x)$ can be obtained in the following way. Let $\mathcal{U}(x_i\,,\theta_\alpha)$ be a graded unitary operator with $\mathcal{U}\mathcal{U}^\ddagger=\mathcal{U}^\ddagger\mathcal{U}=1$. $\mathcal{U}(x_i\,,\theta_\alpha)$ in general can be thought as a 2×2 supermatrix whose entries are functions on $S^{(2,2)}$. $\mathcal{U}(x_i\,,\theta_\alpha)$ acts on $\mathcal{P}_\kappa$ by conjugation and generates a set of supersymmetric extensions $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$:

$$\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)=\mathcal{U}^\ddagger\,\mathcal{P}_\kappa(x)\,\mathcal{U}\,. \tag{5.6}$$

It is easy to see that $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ satisfies $\mathcal{Q}^2_\kappa(x_i\,,\theta_\alpha)=Q_\kappa(x_i\,,\theta_\alpha)$ and $\mathcal{Q}^\ddagger_\kappa(x_i\,,\theta_\alpha)=\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$. Thus $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ is a (super)-projector. The real superfield on $S^{(2,2)}$ associated to $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ is given by

$$\Phi'_a(x_i\,,\theta_\alpha)=\mathrm{Tr}\,\tau_a\mathcal{Q}_\kappa\,. \tag{5.7}$$

In order to perform an indirect check that establishes that $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ are indeed the supersymmetric projectors that reproduces the superfields on $S^{(2,2)}$ subject to

$$\Phi'_a\Phi'_a=1\,, \tag{5.8}$$

we proceed as follows. First we expand $\mathcal{U}(x_i\,,\theta_\alpha)$ in powers of the Grassmann variables as

$$\mathcal{U}(x_i\,,\theta_\alpha)=\mathcal{U}_0(x_i)+C_{\alpha\beta}\theta_\beta\mathcal{U}_\alpha(x_i)+\frac{1}{2}\mathcal{U}_2(x_i)C_{\alpha\beta}\theta_\alpha\theta_\beta \tag{5.9}$$

where $\mathcal{U}_0\,,\mathcal{U}_\alpha(\alpha=\pm)$ and $\mathcal{U}_2$ are all 2×2 graded unitary matrices. The requirement that $\mathcal{U}(x_i\,,\theta_\alpha)$ is graded unitary makes $\mathcal{U}_0(x_i)$ unitary, whereas $\mathcal{U}_\alpha(x_i)$ are uniquely determined by $\mathcal{U}_\alpha(x_i)=H_\alpha(x_i)\mathcal{U}_0(x_i)$ where H_α are 2×2 odd supermatrices with the reality condition $H^\ddagger_\alpha=-C_{\alpha\beta}H_\beta$. Moreover, with the ansatz that $\mathcal{U}_2=A\mathcal{U}_0$ with A being an arbitrary 2×2 even supermatrix, graded unitarity of $\mathcal{U}(x_i\,,\theta_\alpha)$ further restricts the symmetric part of A as:

$$A+A^\dagger=-C_{\alpha\beta}H_\alpha H_\beta\,. \tag{5.10}$$

Using the expansion (5.9) in (5.6) and subsequently the resulting expression in (5.7) together with the properties listed above it is straightforward to extract the component fields of the superfield $\Phi'_a(x_i\,,\theta_\alpha)$. We find

$$\begin{aligned} n^{\kappa\prime}_a &:= \mathrm{Tr}\,\tau_a U_0^\dagger\mathcal{P}_\kappa U_0\,, & (5.11)\\ \psi^{a\prime}_\alpha &:= -2i(\vec{n}^{\kappa\prime}\times\vec{H}'_\alpha)^a=\mathrm{Tr}\,\tau_a U_0^\dagger[H_\alpha\,,\mathcal{P}_\kappa]U_0\,, & (5.12)\end{aligned}$$

and after using (5.10) that

$$\begin{aligned} F'_a &:= 4(\vec{H}'_+\cdot\vec{H}'_-)n^{\kappa\prime}_a-2\vec{H}^{a\prime}_+(\vec{n}^{\kappa\prime}\cdot\vec{H}'_-)-(\vec{n}^{\kappa\prime}\cdot\vec{H}'_+)2\vec{H}^{a\prime}_-+i(\vec{n}^{\kappa\prime}\times(\vec{A}'-\vec{A}^{\dagger\prime}))^a\\ &= \mathrm{Tr}\,\tau_a U_0^\dagger(\mathcal{P}_\kappa A+A^\dagger\mathcal{P}_\kappa-C_{\alpha\beta}H_\beta\mathcal{P}_\kappa H_\alpha)U_0 \qquad (5.13)\end{aligned}$$

where $\vec{H}^{a\prime}_\alpha=H^{a\prime}_\alpha\tau^a$ and $\vec{A}^{a\prime}=A^{a\prime}\tau^a$. By direct computation from above and using (5.8) we find

$$n^{\kappa\prime}_a n^{\kappa\prime}_a=1\,,\qquad n^{\kappa\prime}_a F'_a=\frac{1}{2}\psi^{\ddagger\prime}_a\psi'_a\,,\qquad n^{\kappa\prime}_a\psi^{a\prime}_\pm=0\,. \tag{5.14}$$

Comparing (5.14) with (5.5) we observe that they are identical. Therefore we conclude that the superfield associated to the super-projector $\mathcal{Q}_\kappa$ is the same as the superfield in supersymetric non-linear sigma model of the previous subsection.

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5.3 SUSY action revisited

We are now ready to give the formulation of non-linear sigma model on the supersphere using the (super)-projectors. In close analogy with the $\mathbb{C}P^1$ case the supersymmetric action in (5.2) with the constraint (5.3) translates to

$$S_\kappa^{\mathrm{SUSY}} = \frac{1}{2\pi}\int d\mu\, \mathrm{Tr}\left[(d_\alpha \mathcal{Q}_\kappa)(d_\alpha \mathcal{Q}_\kappa) + \frac{1}{4}(\gamma \mathcal{Q}_\kappa)(\gamma \mathcal{Q}_\kappa)\right] \tag{5.15}$$

The even part of this action, as well as the one given in (5.2) is nothing but the action S_κ of the $\mathbb{C}P^1$ theory given in (3.2) and (3.1), respectively. In other words, the action S_κ^{SUSY} is the supersymmetric extension of S_κ on S^2 to $S^{(2,2)}$. Thus in the supersymmetric theory it is possible to interpret the index κ carried by the action as the winding number of the corresponding $\mathbb{C}P^1$ theory.

We recall that d_α and γ are both derivations in the superalgebra osp$(2,2)$. Therefore they obey a graded Leibnitz rule and from $\mathcal{Q}_\kappa^2 = \mathcal{Q}_\kappa$ we find

$$\mathcal{Q}_\kappa d_\alpha \mathcal{Q}_\kappa = d_\alpha \mathcal{Q}_\kappa(\mathbf{1} - \mathcal{Q}_\kappa)\,. \tag{5.16}$$

This enables us to write

$$\mathrm{Tr}\, d_\alpha \mathcal{Q}_\kappa(\mathbf{1} - \mathcal{Q}_\kappa) d_\alpha \mathcal{Q}_\kappa = \mathrm{Tr}(\mathbf{1} - \mathcal{Q}_\kappa)(d_\alpha \mathcal{Q}_\kappa)^2 = \frac{1}{2}\mathrm{Tr}(d_\alpha \mathcal{Q}_\kappa)^2\,. \tag{5.17}$$

Equations (5.16) and (5.17) continue to hold when d_α is replaced by γ as well. The action then takes the form

$$S_\kappa^{\mathrm{SUSY}} = \frac{1}{\pi}\int d\mu\, \mathrm{Tr}\left[\mathcal{Q}_\kappa(d_\alpha \mathcal{Q}_\kappa)(d_\alpha \mathcal{Q}_\kappa) + \frac{1}{4}\mathcal{Q}_\kappa(\gamma \mathcal{Q}_\kappa)(\gamma \mathcal{Q}_\kappa)\right]. \tag{5.18}$$

It is possible that this form of the action could play an important role for obtaining an supersymmetric generalization of the BPS equation since an analogues expression in the $\mathbb{C}P^1$ case [1] have been employed to achieve this result.

6. Fuzzy projectors and sigma models

6.1 Fuzzy $\mathbb{C}P^1$ model

In [1] the $\mathbb{C}P^1$ model has been quantized as follows. Let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{0\}$. In terms of ξ we define

$$z = \frac{\xi}{|\xi|}\,, \qquad |\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}\,, \qquad x_i = z^\dagger x_i z\,. \tag{6.1}$$

ξ_α and $\bar{\xi}_\alpha$ are quantized by replacing them with a pair of annihilation a_α and creation $a_\alpha^\dagger$ operators respectively. With this substitution $|\xi|$ becomes the square root of the number operator and we have

$$\begin{aligned}
\hat{N} &= \hat{N}_1 + \hat{N}_2\,, \qquad \hat{N}_1 = a_1^\dagger a_1\,, \qquad N_2 = a_2^\dagger a_2 \\
\hat{z}_\alpha^\dagger &= \frac{1}{\sqrt{\hat{N}}} a_\alpha^\dagger = a_\alpha^\dagger \frac{1}{\sqrt{\hat{N}+1}}\,, \qquad \hat{z}_\alpha = \frac{1}{\sqrt{\hat{N}+1}} a_\alpha = a_\alpha \frac{1}{\sqrt{\hat{N}}}\,, \\
\hat{x}_i &= \frac{1}{\hat{N}} a^\dagger \tau_i a\,.
\end{aligned} \tag{6.2}$$

In the light of this conjecture it is easy to see that the quantized version of the partial isometry $\vartheta_\kappa^\dagger$ defined in (2.4) and its hermitean conjugate reads

$$\begin{aligned}\hat{\vartheta}_\kappa^\dagger &= \frac{1}{\sqrt{\hat{Z}_\kappa}}\left(a_1^{\dagger\kappa} \quad a_2^{\dagger\kappa} \right), \qquad \hat{\vartheta}_\kappa = \begin{pmatrix} a_1^\kappa \\ a_2^\kappa \end{pmatrix}\frac{1}{\sqrt{\hat{Z}_\kappa}}, \qquad \hat{\vartheta}_\kappa^\dagger\hat{\vartheta}_\kappa = \mathbf{1}, \\ \hat{Z}_\kappa &= \hat{Z}_\kappa^{(1)} + \hat{Z}_\kappa^{(2)}, \qquad \hat{Z}_\kappa^{(\alpha)} = \hat{N}_\alpha(\hat{N}_\alpha - 1)\cdots(\hat{N}_\alpha - \kappa + 1). \end{aligned} \tag{6.3}$$

The fuzzy analogue of (2.6) can now be written as

$$\hat{P}_\kappa(x) = \hat{\vartheta}_\kappa\hat{\vartheta}_\kappa^\dagger = \begin{pmatrix} a_1^\kappa \frac{1}{\hat{Z}_\kappa} a_1^{\dagger\kappa} & a_1^\kappa \frac{1}{\hat{Z}_\kappa} a_2^{\dagger\kappa} \\ a_2^\kappa \frac{1}{\hat{Z}_\kappa} a_1^{\dagger\kappa} & a_2^\kappa \frac{1}{\hat{Z}_\kappa} a_2^{\dagger\kappa} \end{pmatrix} \tag{6.4}$$

where for example

$$a_1^\kappa \frac{1}{\hat{Z}_\kappa} = \frac{1}{(\hat{N}_1+\kappa)\cdots(\hat{N}_1+1) + \hat{Z}_\kappa^{(2)}} a_1^\kappa, \qquad a_1^\kappa a_1^{\dagger\kappa} = (\hat{N}_1+\kappa)\cdots(\hat{N}_1+1). \tag{6.5}$$

The unitary matrix U introduced to generate all possible projectors $\mathcal{P}_\kappa$ from P_κ also have fuzzy analogue. It is a 2×2 unitary matrix $\hat{U}$, with matrix entries being polynomials in $a_\alpha^\dagger a_\beta$. Thus the most general fuzzy projectors are

$$\hat{\mathcal{P}}_\kappa = \hat{U}\hat{P}_\kappa\hat{U}^\dagger. \tag{6.6}$$

From (6.4) it is clear that $\hat{\mathcal{P}}_\kappa$ acts in general on $\mathcal{F}^2 := \mathcal{F}\otimes\mathbb{C}^2$ where $\mathcal{F}$ stands for the standard Fock space. It also follows from (6.4) that $\hat{P}_\kappa$ commutes with the number operator $\hat{N}$, as can be checked directly. Consequently, we can restrict ourselves to work on a finite dimensional subspace $\mathcal{F}_n$ of dimension $n+1$ of $\mathcal{F}$. Then $\hat{\mathcal{P}}_\kappa$ act on the finite dimensional Hilbert space $\mathcal{F}_n^2 := \mathcal{F}_n\otimes\mathbb{C}^2$ and this allows one to formulate a finite dimensional matrix model for projectors $\hat{\mathcal{P}}_\kappa$.

In [1] this has been done to write down the fuzzy $\mathbb{C}P^1$ model. They found that the fuzzy action corresponding to (3.2) is

$$S_{F,\kappa} = \frac{1}{4\pi}\frac{1}{2(n+1)}\,\mathrm{Tr}_{\hat{N}=n}(\mathcal{L}_i\hat{\mathcal{P}}_\kappa)(\mathcal{L}_i\hat{\mathcal{P}}_\kappa) \tag{6.7}$$

where $\mathcal{L}_i\hat{\mathcal{P}}_\kappa = [L_i\,,\hat{\mathcal{P}}_\kappa]$ and the trace is over $\mathcal{F}_n^{(2)}$.

6.2 Fuzzy supersymmetric model

In much the same way the supersymmetric projectors $\mathcal{Q}_\kappa$ have been constructed from $\mathcal{P}_\kappa$ in section 5, we can construct supersymmetric extensions of $\hat{\mathcal{P}}_\kappa$ by the graded unitary transformation

$$\hat{\mathcal{Q}}_\kappa = \hat{\mathcal{U}}^\ddagger\hat{\mathcal{P}}_\kappa\hat{\mathcal{U}} \tag{6.8}$$

where $\hat{\mathcal{U}}$ is a 2×2 supermatrix whose entries are polynomials in $a_\alpha^\dagger a_\beta$ and $b^\dagger b$ and where b and $b^\dagger$ are fermionic annihilation and creation operators with the anti-commutation relation $\{b\,,b^\dagger\} = 1$.

$\widehat{\mathcal{Q}}_\kappa$ defined (6.8) acts on the finite dimensional space $\tilde{\mathcal{F}}_n^2 = \tilde{\mathcal{F}}_n \otimes \mathbb{C}^2$. Here $\tilde{\mathcal{F}}$ is the $n+1$ dimensional subspace of the Hilbert space $\tilde{\mathcal{F}}$ spanned by the kets $|n_1, n_2, n_3\rangle$ where n_3 labels the fermionic part taking on the values 0 and 1 only. It is readily seen that $\widehat{\mathcal{Q}}_\kappa$ commutes with the supersymmetric number operator $\hat{\mathcal{N}} = a_\alpha^\dagger a_\alpha + b^\dagger b$. In close analogy with the fuzzy $\mathbb{C}P^1$ model, it is now possible to write down a finite dimensional (super)matrix model for the (super)-projectors $\widehat{\mathcal{Q}}_\kappa$.

Making use of (5.1) once more the action for the fuzzy supersymmetric model is given as

$$S_{F,\kappa}^{\mathrm{SUSY}} = \frac{1}{2\pi} \mathrm{Str}_{\hat{\mathcal{N}}=n} \left[(\mathcal{D}_\alpha \widehat{\mathcal{Q}}_\kappa)(\mathcal{D}_\alpha \widehat{\mathcal{Q}}_\kappa) + \frac{1}{4} (\Xi \widehat{\mathcal{Q}}_\kappa)(\Xi \widehat{\mathcal{Q}}_\kappa) \right], \tag{6.9}$$

where $\mathcal{D}_\alpha \widehat{\mathcal{Q}}_\kappa = \{D_\alpha \, , \widehat{\mathcal{Q}}_\kappa\}$ and $\Xi \widehat{\mathcal{Q}}_\kappa = [\Gamma \, , \widehat{\mathcal{Q}}_\kappa]$. "Str" in the above expression is the supertrace over the finite dimensional space $\tilde{\mathcal{F}}_n^2$. Obviously, in the large $\hat{\mathcal{N}} = n$ limit (6.9) approximates the action given in (5.15).

7. Conclusions

In this paper we have obtained the fuzzy version of supersymmetric non-linear sigma model on $S^{(2,2)}$. Our approach has utilized the use of supersymmetric extensions of the Bott projectors and generalized results of $\mathbb{C}P^1$ model to supersymmetric theories. A natural question to be addressed is the supersymmetric generalization of the BPS equation. We hope to report any development on this issue elsewhere.

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